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OPTIMAL DESIGN OF ELASTIC BEAMS UNDER MULTIPLE DESIGN CONSTRAINTS

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Abstract—This paper discusses the optimization of elastic beams under multiple load conditions and self-weight subject to stress and displacement constraints as well as limits on the cross-sectional area and its rate of spatial change ("Niordson constraint"). The general formulation allows for the effect of both bending moments and shear forces on the stresses and deflections. The proposed method is based on *static–kinematic optimality criteria* which have been successfully used in optimal plastic design. In the above approach, the Lagrangian of the equilibrium condition is regarded as an "associated" (or "Pragerian") displacement field. The general theory is then illustrated with the example of a built-in beam subjected to stress and Niordson constraints; the statical redundancy of the beam provides a (zero) displacement constraint. Allowance is also made for the cost of clamping moments. It is found that, in general, some segments of the beams are "understressed" and the associated displacement field contains concentrated rotations ("curvature impulses"). Moreover, the solution of this example is found to take on a surprising number of different forms. A beam example with allowance for self-weight will be discussed in Part II of this study.

INTRODUCTION

The main aim of this paper is to extend the approach based on static–kinematic optimality criteria from plastically designed beams to elastic beams under a variety of design constraints. Whereas the idea of optimality criteria in plastic beam design was already used by Heyman[1] in the 1950s and explored more systematically by Prager and Shield (e.g. Ref. [2]) in the 1960s, it was employed in elastic beam design only more recently. The potential complexities of this latter application will be demonstrated through an example in this paper.

The beams under consideration are horizontal and statically indeterminate. They are subjected to several alternate vertical load systems besides their own weight (dead load).

In elastic beam design we may consider various design constraints including the following.

(1) *Deflection constraints* requiring the deflections at prescribed points not to exceed specified values.

(2) *Stress constraints* prescribing the maximum permissible stress values. In calculating the stresses, the effect of both bending moment and shear force on a cross-section may be taken into consideration.

(3) *Constraints on the minimum and maximum cross-sectional area.*

(4) "*Niordson constraints*" limiting the maximum spatial rate of change of cross-sectional dimensions. This idea was introduced by Niordson[3, 4] in the context of plates for which he proposed a restriction on the slope of the plate thickness (or "taper"). As he correctly pointed out, this type of constraint ensures that the resulting design does not contain (i) sudden changes in the cross-section and thus satisfies the original assumptions of the underlying theory (e.g. plate or beam theory); nor (ii) points at which the cross-sectional area vanishes. The use of Niordson constraints in plastic design was recently discussed in Ref. [5].

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As the literature on the optimal design of elastic beams is quite extensive, only a few selected papers will be reviewed herein. An optimality criterion for statically determinate elastic beams with a prescribed *deflection* was proposed in 1961 by Barnett[6] and his approach was generalized ten years later by Prager[7]. In 1973, Masur[8] proposed a condition for the optimal *location* of hinges and discontinuities in the cross-sectional area[9] in elastic beams with prescribed deflection and this condition was extended to any combination of supports, hinges and discontinuities in the cross-section by Mróz and Rozvany[10].

The minimization of the maximum elastic deflection along the entire length of a beam was discussed in Refs [11, 12].

The optimal design of elastic structures for *stress constraints* was explored by Masur[13] and Taylor[14]. A summary of optimality conditions for elastic beams was given in 1976 in Ref. [15] and later Refs [16, 17] showed that the optimal solution for elastic statically indeterminate beams is *not* everywhere fully stressed if either the cost of supports is taken into consideration or the specific cost function is not symmetrical. References [18–20] discussed the stress design of beams, beam-columns and frames, taking the effect of both bending and shear into consideration. The problem of optimizing elastic beams in the presence of self-weight was investigated in Ref. [21]. A comprehensive theory of optimal segmented elastic structures with stress and deflection constraints was developed recently by three of the authors[22].

PROBLEM FORMULATION

Let the design values of the cross-sectional area, the specific cost (cost per unit length), the flexural beam stiffness and the shear stiffness be $\bar{\alpha}(x)$, $\psi(\bar{\alpha})$, $S(\bar{\alpha})$ and $S_1(\bar{\alpha})$, where x is the distance measured along the beam axis. Furthermore, let the maximum permissible rate of change of the cross-sectional area, the upper and lower limits on the cross-sectional area be θ, α_a and α_b, respectively. The cross-sectional area requirement for a given bending moment M_j and shear force M'_j (equilibrating the jth load condition) is denoted by $\alpha = \alpha(M_j, M'_j)$, where a prime denotes differentiation with respect to x. It is assumed that the shape of the cross-section is restricted in a manner that ensures a unique functional relationship among $\alpha = \alpha(M_j, M'_j)$, $S(\bar{\alpha})$ and $S_1(\bar{\alpha})$, after local optimization (if necessary). A more general formulation is discussed in Section 1.10 and Chap. 6 in Ref. [23]. The beam is subject to m alternate loading conditions p_j ($j = 1, 2, \ldots, m$) and limits d_i ($i = 1, 2, \ldots, n$) are prescribed on deflections at n points. It is important to note that a deflection constraint can either be an operational one or a physical one (representing a redundancy). If the latter is due to a rigid support then $d_i = 0$ (where d_i may signify a deflection or a rotation). Using the above notation, a Niordson constraint can be expressed as

$$|\bar{\alpha}'| \leqslant \theta \tag{1}$$

or

$$\bar{\alpha}' - \theta + s_1(x) = 0, \quad -\bar{\alpha}' - \theta + s_2(x) = 0 \tag{2}$$

where $s_1(x)$ and $s_2(x)$ are non-negative slack functions. Similarly, the minimum and maximum cross-sectional area constraints become

$$-\bar{\alpha} + \alpha_a + s_3(x) = 0, \quad \bar{\alpha} - \alpha_b + s_4(x) = 0 \tag{3}$$

where $s_3(x)$ and $s_4(x)$ are slack functions. The equilibrium state of elastic beams is described by the relation

$$M''_j + p_j + \rho\bar{\alpha} = 0 \quad (j = 1, 2, \ldots, m) \tag{4}$$

where ρ is the specific weight of the beam material. The stress constraints then become

$$-\bar{\alpha}+\alpha(M_j, M'_j)+s_j(x) = 0 \quad (j = 1,2,\ldots,m). \tag{5}$$

Denoting by m_i the beam moments caused by a unit "dummy" load at the prescribed location of the deflection limit d_i we have the deflection constraints

$$\int_D \left(\frac{M_j m_i}{S(\bar{\alpha})} + \frac{M'_j m'_i}{S_1(\bar{\alpha})}\right) \mathrm{d}x - d_i - s_{ji} = 0 \quad (i = 1,2,\ldots,n; j = 1,2,\ldots,m) \tag{6}$$

where s_{ji} are positive slack variables. For physical deflection constraints (representing rigid redundant supports), we have $d_i = s_{ji} = 0$ since such constraints must always be satisfied as an equality.

Introducing the Lagrangian functions $\lambda_1(x)$, $\lambda_2(x)$, $\lambda_3(x)$, $\lambda_4(x)$, $\omega_j(x)$ and $u(x)$ and the Lagrangian multipliers v_{ij}, the above problem can be stated as follows:

$$\min \Phi = \int_D \Big\{\psi(\bar{\alpha}) + \lambda_1(\bar{\alpha}' - \theta + s_1) + \lambda_2(-\bar{\alpha}' - \theta + s_2) + \lambda_3(-\bar{\alpha} + \alpha_a + s_3)$$

$$+ \lambda_4(\bar{\alpha} - \alpha_b + s_4) + \sum_j \omega_j[-\bar{\alpha} + \alpha(M_j, M'_j) + s_j] + \sum_j u_j(M''_j + p_j + \rho\bar{\alpha})\Big\} \mathrm{d}x$$

$$+ \sum_i \sum_j v_{ij}\left\{\int_D [M_j m_i/S(\bar{\alpha}) + M'_j m'_i/S_1(\bar{\alpha})] \,\mathrm{d}x - d_i - s_{ji}\right\} \tag{7}$$

where Φ is the total "cost" to be minimized and D is the "structural domain" (beam axis between beam ends, $D = [x : 0 \leqslant x \leqslant L]$).

DERIVATION OF OPTIMALITY CONDITIONS

Necessary minimality conditions for variations of $\bar{\alpha}$, M_j and the slack functions/variables furnish (see, e.g. pp. 18–21 and 25 of Ref. [15])

$$\sum_j \omega_j = \psi_{,\bar{\alpha}} + \sum_j \rho u_j - \lambda'_1 + \lambda'_2 - \lambda_3 + \lambda_4 - \sum_i \sum_j v_{ij}[m_i M_j S_{,\bar{\alpha}}/S^2 + m'_i M'_j S_{1,\bar{\alpha}}/S_1^2] \tag{8}$$

$$-u''_j = \omega_j[\alpha_{,M_j} - (\alpha_{,M'_j})'] + \sum_i v_{ji}[m_i/S - (m'_i/S_1)'] \tag{9}$$

$$\begin{array}{lll} (\text{for } s_i(x) > 0)\ \lambda_i = 0, & (\text{for } s_i(x) = 0)\ \ \lambda_i \geqslant 0 & (i = 1,2,3,4) \\ (\text{for } s_j(x) > 0)\ \omega_j = 0, & (\text{for } s_j(x) = 0)\ \ \omega_j \geqslant 0 & (j = 1,2,\ldots,m) \\ (\text{for } s_{ij} > 0)\ v_{ij} = 0, & (\text{for } s_{ij} = 0)\ \ v_{ij} \geqslant 0 & (i = 1,2,\ldots,n; j = 1,2,\ldots,m) \end{array} \tag{10}$$

where a comma followed by a subscript denotes partial differentiation with respect to that variable.

Moreover, transversality conditions (e.g. pp. 21–23 of Ref. [15]) furnish the same end conditions for u_j as if $u_j(x)$ were a deflection field for an elastic beam (e.g. $u_j = u'_j = 0$ at built-in ends).

This means that, in analogy with the Prager–Shield theory of plastic optimal design[2], $u_j(x)$ can be interpreted as an "associated" or "Pragerian" beam deflection for the jth load condition and $-u''_j = \kappa_j$ as the beam curvature. As in plastic design for bending and shear (Ref. [15] and pp. 51–53 of Ref. [24]) $\omega_j\alpha_{,M_j}$ then becomes the "generalized" flexural strain (= curvature caused by flexure) and $\omega_j\alpha_{,M'_j}$ is the "generalized" shear strain. Naturally the first derivative of the latter also contributes to the total curvature κ_j. However, the second term on the right-hand side of eqn (9) occurs only in optimal *elastic* designs.

APPLICATION TO BEAMS OF CONSTANT DEPTH

In this paper, the above theory will be applied to a special case in which:

(a) the total cost is the beam weight (or volume) such that the specific cost becomes $\psi(\bar{\alpha}) = \bar{\alpha}$;

(b) there is only one loading condition ($m = 1$) and one deflection constraint ($n = 1$), and hence subscripts i and j may be omitted;

(c) the beam has a given depth and variable width and the effect of the shearing force is neglected, such that $\alpha(M, M') = k|M|$, $S(\bar{\alpha}) = a\bar{\alpha}$ where k and a are given constants;

(d) the self-weight is neglected, $\rho = 0$.

Then eqns (8) and (9) reduce to

$$\omega = 1 - \lambda_1' + \lambda_2' - \lambda_3 + \lambda_4 - vamM/S^2 \tag{11}$$

$$-u'' = \omega k \operatorname{sgn} M + vm/S. \tag{12}$$

It is assumed that the deflection constraint is active, i.e. it is satisfied as an equality by the optimal solution. Then the following types of segments may occur in the solution.

(a) *Regions governed by the stress constraint*: R_s

Since for this region $\bar{\alpha} = k|M| = kM \operatorname{sgn} M$, $S = a\bar{\alpha} = akM \operatorname{sgn} M$, and $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \equiv \lambda_4 = 0$, eqns (11) and (12) reduce to

$$-u'' = k \operatorname{sgn} M(1 - vamM/S^2) + vm/S = k \operatorname{sgn} M \tag{13}$$

as in plastic design[15] for a specific cost function $k|M|$. This means that over fully stressed segments the associated curvature has a constant magnitude and its sign is the same as that of the bending moment.

(b) *Regions governed by the minimum area constraint*: R_a

For these regions $\bar{\alpha} = \alpha_a$, $s_3 \equiv 0$, $\omega \equiv \lambda_1 \equiv \lambda_2 \equiv \lambda_4 \equiv 0$ and hence we have

$$-u'' = vam/S. \tag{14}$$

(c) *Regions governed by the Niordson constraint*: R_N^+ *and* R_N^- (*for* $\bar{\alpha}' = 0$ *and* -0)

For R_N^+ regions, $\lambda_1 \neq 0$, $\lambda_2 \equiv \lambda_3 \equiv \lambda_4 \equiv \omega \equiv 0$ and hence eqn (14) again holds. Moreover, eqn (11) furnishes

$$\lambda_1' = 1 - vamM/S^2. \tag{15}$$

Relations (14) and (15) also apply to R_N^- regions (with $-\lambda_2'$ replacing λ_1' in eqn (15)).

(d) *Regions governed by the deflection constraint* (*understressed regions*): R_d

If the optimal solution consisting of the types of regions under (a)–(c) violates the deflection constraint, then in some regions the beam must have a cross-sectional area greater than the one required by stress, minimum area and Niordson constraints. In this case $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \equiv \lambda_4 \equiv \omega \equiv 0$ and hence eqn (14) again holds. Moreover, eqn (11) implies

$$vamM/S^2 = 1, \quad S = a\bar{\alpha} = \sqrt{(vamM)} \tag{16}$$

which is identical with the optimality criterion of Barnett[6] and Prager[7]. Note that in the R_d regions vm and M must have the same sign.

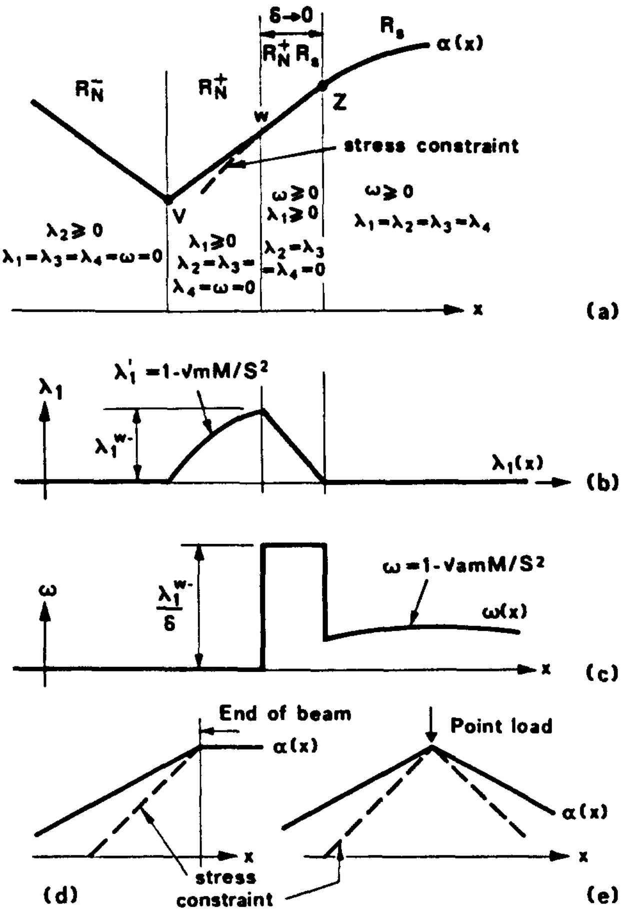

Fig. 1. Transversality conditions for the Lagrangians at region boundaries.

(e) *Regions governed by the maximum area constraint*: R_b

These regions are likely to be isolated points because in general the constraint $\bar{\alpha} \leqslant \alpha_b$ permits $\bar{\alpha} = \alpha_b$ only locally. If an R_b-type point falls within an R_s region, for example, then it is also fully stressed, giving $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \equiv 0$, $\lambda_4 \geqslant 0$, $\omega \geqslant 0$ and hence eqns (11) and (12) furnish

$$-u'' = k \operatorname{sgn} M(1+\lambda_4), \quad \lambda_4 \geqslant 0 \tag{17}$$

giving an indefinitely large curvature which can be a concentrated rotation (curvature impulse).

(f) *Optimality conditions for region boundaries*

It will be shown in this section that at most region boundaries the Lagrangians $\lambda_1(x)$ and $\lambda_2(x)$ are continuous and have a zero value. However, a step in the Lagrangian $\lambda_1(x)$ (or $\lambda_2(x)$) and an impulse (Dirac distribution) in both the Lagrangian $\omega(x)$ and the curvature $-u''(x)$ can occur if:

(i) the region boundary is between regions $R_N^+/R_s(R_N^-/R_s)$ and
(ii) the slope $\alpha'(x)$ is continuous across the boundary under consideration.

The reasons for the above conclusions are explained in Fig. 1 in which an R_N^+ region is adjacent to an R_N^- and an R_s region. The respective values of the Lagrangians for various regions are also indicated in Fig. 1(a).

It is first assumed that between the R_N^+ and the R_s regions a narrow region of width δ occurs and that both the stress constraint and the Niordson constraint are active over this "$R_N^+ R_s$" region. Then a limiting process is carried out for $\delta \to 0$ and hence the original region layout ($W \to Z$ in Fig. 1(a)) is restored. It follows from eqns (10) that over the $R_N^+ R_s$ region $\omega \geqslant 0$, $\lambda_1 \geqslant 0$ but the other Lagrangians are zero.

At the boundary of R_N^+/R_N^- regions (Point V in Fig. 1) the usual transversality conditions (pp. 21–23 of Ref. [15]) for the variation of α' imply $-\lambda_2^{V-} = \lambda_1^{V+}$ where "$-$" and "$+$" indicate that we approach the region boundary from the left and from the right, respectively. Then the conditions $\lambda_1 \geqslant 0$ and $\lambda_2 \geqslant 0$ imply

$$\lambda_2^V = \lambda_1^V = 0. \tag{18a}$$

Moreover, the same transversality conditions for the variation of α' at the boundary of $R_N^+ R_s/R_s$ regions (Point Z in Fig. 1(a)) imply

$$\lambda_1^Z = 0. \tag{18b}$$

By eqn (15) the slope of λ_1 in the R_N^+ region is in general nonzero and hence λ_1 in general takes on a non-zero value at the boundary of $R_N^+/R_N^+ R_s$ regions (W in Fig. 1(a)). Over the $R_N^+ R_s$ region eqn (11) reduces to

$$\omega = 1 - \lambda_1' - vamM/S^2 \tag{19}$$

in which ω and λ_1' approach infinity when $\delta \to 0$ (Fig. 1) and hence other terms can be neglected. It follows that the integral of the ω-impulse between W and Z is λ_1^{W-} (Fig. 1) which by eqns (15) and (18) is given by

$$\int_W^Z \omega \, dx = \int_{R_N^+} (1 - vamM/S^2) \, dx. \tag{20a}$$

A similar impulse occurs at the common boundary of R_N^- and R_s regions but it has the magnitude

$$\int_{R_N^-} (vamM/S^2 - 1) \, dx. \tag{20b}$$

Note: If the R_s region is restricted to a point (Figs 1(d) and (e)) then eqn (20a) or (20b) may apply even when the slope of the stress constraint does not equal the slope of the specific cost function $\alpha(x)$. This can be shown by a limiting process in which a concentrated reaction or point load is temporarily replaced by a distributed load over a small beam length, making the two slopes equal at the ends of such a length.

For all R_N^+ (and R_N^-) regions which are not adjacent to an R_s region we have $\lambda_1 = 0$ ($\lambda_2 = 0$) at both boundaries and hence an additional optimality condition must be satisfied

$$\int_{R_N^+} (1 - vamM/S^2) \, dx = 0, \quad \int_{R_N^-} (vamM/S^2 - 1) \, dx = 0. \tag{20c}$$

The curvature impulses described in this section are similar to those found in plastic design with Niordson constraints[5].

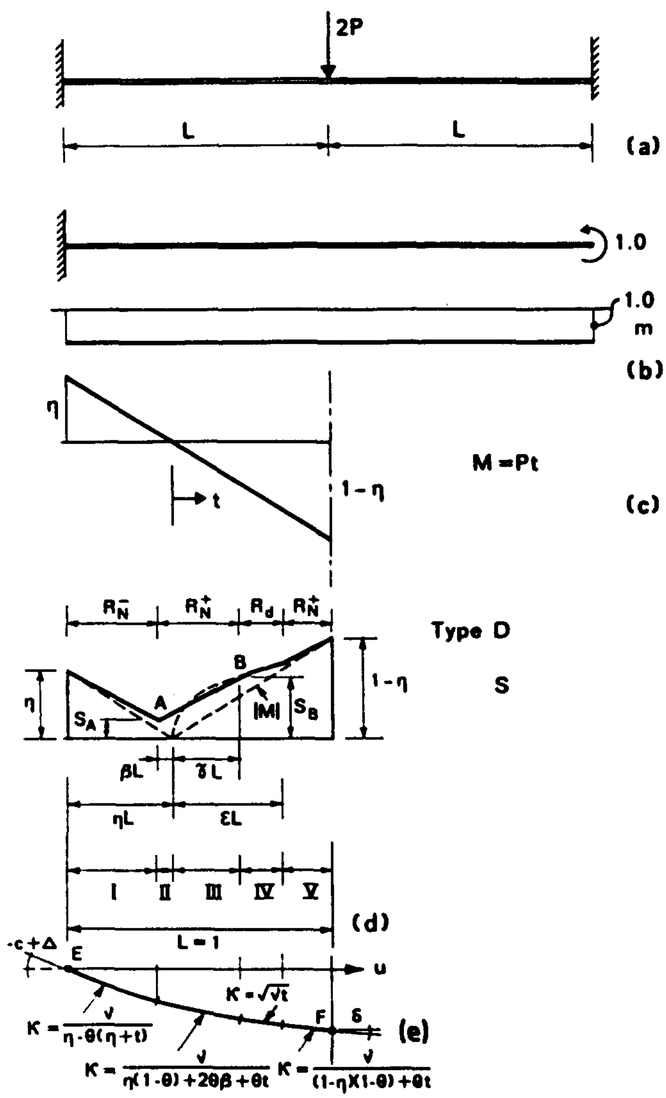

Fig. 2. Beam example and Type D regions.

EXAMPLE: BUILT-IN BEAM WITH A POINT LOAD

Consider a beam of given depth and variable width with the support and loading conditions as shown in Fig. 2(a). For this and similar beams $\alpha = k|M|$ and $S = a\bar{\alpha}$. However, after suitable nondimensionalization we may set $a = k = 1$ and $L = P = 1$ (Fig. 2(a)). The cost of clamping moments will be $c|M|$ where c is a constant.

For expected symmetric solutions, the only kinematic requirement is that the slope vanishes at both ends. This can be assured by adopting the unit "dummy" load and the corresponding moment diagram $m(x)$ of Fig. 2(b) and introducing the deflection constraint

$$\int_{-\eta}^{1-\eta} (Mm/S)\,\mathrm{d}t = 0 \tag{21a}$$

where the origin of the coordinate t and the moments caused by the external load are shown in Fig. 1(c). It will be proved below that the optimal solution can take several forms. The ranges of validity of these solutions are shown in Fig. 3. First, one of the most complicated region patterns will be considered from which several other types of solutions can be derived.

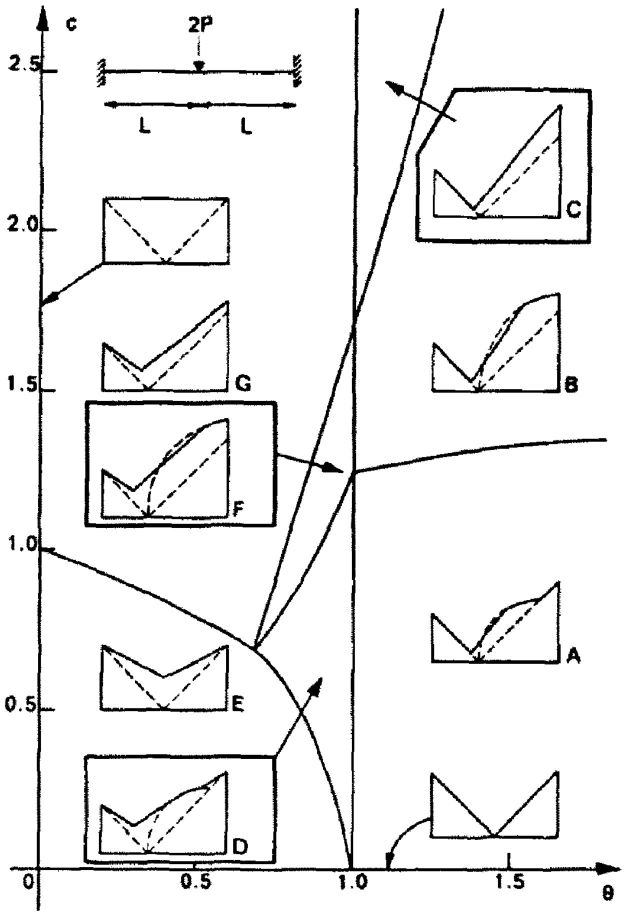

Fig. 3. Range of validity of various types of optimal regions.

Type D *solutions*

This type of solution consists of one R_N, one R_d and two R_N^+ type regions (Fig. 2(d)). It can easily be shown that

$$S_A = \eta(1-\theta)+\theta\beta$$
$$S_B = \eta(1-\theta)+(2\beta+\gamma)\theta. \qquad (21b)$$

Kinematic admissibility. The first useful condition is provided by kinematic requirement (21a). We shall summarize the subtotals of this integral for segments I–V in Fig. 2(d).

I. (R_N^-) $$\int_{-\eta}^{-\beta}(M/S)\,\mathrm{d}t = \int_{-\eta}^{-\beta}\{t/[\eta(1-\theta)-\theta t]\}\,\mathrm{d}t$$
$$= (\beta-\eta)/\theta-[\eta(1-\theta)\ \ln\ (1-\theta+\theta\beta/\eta)]/\theta^2. \qquad (22)$$

II and III. (R_N^+) $$\int_{-\beta}^{\gamma}(M/S)\,\mathrm{d}t = \int_{-\beta}^{\gamma}\{t/[\eta(1-\theta)+2\theta\beta+\theta t]\}\,\mathrm{d}t$$
$$= (\gamma+\beta)/\theta-\{[\eta(1-\theta)+2\theta\beta]/\theta^2\}$$
$$\times \ln\ \{[\eta(1-\theta)+2\theta\beta+\theta\gamma]/[\eta(1-\theta)+\theta\beta]\}. \qquad (23)$$

IV. (R_d) By eqns (16)

$$S = \sqrt{(vt)}. \tag{24}$$

However, at $t = \varepsilon$ (Fig. 2(d))

$$S(\varepsilon) = (1-\eta)-\theta(1-\eta-\varepsilon). \tag{25}$$

Then from eqns (24) and (25) we have

$$v = [(1-\eta)(1-\theta)+\theta\varepsilon]^2/\varepsilon, \quad S = [(1-\eta)(1-\theta)+\theta\varepsilon](t/\varepsilon)^{1/2}. \tag{26}$$

Moreover, at $t = \gamma$ the continuity of $S(t)$ requires (see eqns (21b) and (24))

$$S_B = \sqrt{(v\gamma)}, \quad v = [\eta(1-\theta)+2\theta\beta+\theta\gamma]^2/\gamma \tag{27a}$$

$$[(1-\eta)(1-\theta)+\theta\varepsilon](\gamma/\varepsilon)^{1/2} = \eta(1-\theta)+2\theta\beta+\theta\gamma \tag{27b}$$

$$\int_\gamma^\varepsilon (M/S)\,dt = \left\{\int_\gamma^\varepsilon (t/\sqrt{(vt)})\,dt = [2/(3\sqrt{v})](\varepsilon^{3/2}-\gamma^{3/2})\right\}$$

$$= \{(2/3)\gamma^{1/2}/[\eta(1-\theta)+2\theta\beta+\theta\gamma]\}(\varepsilon^{3/2}-\gamma^{3/2}). \tag{28}$$

V. (R_N^+)

$$\int_\varepsilon^{1-\eta} (M/S)\,dt = \int_\varepsilon^{1-\eta} \{t/[(1-\eta)(1-\theta)+\theta t]\}\,dt$$

$$= [(1-\eta-\varepsilon)/\theta]+[(1-\eta)(1-\theta)/\theta^2]\ln[1-\theta+\theta\varepsilon/(1-\eta)]. \tag{29}$$

Condition (21a) can then be expressed by adding the right-hand sides of eqns (22), (23), (28) and (29) and making the sum equal to zero. Denoting this sum by $G(\beta,\gamma,\varepsilon,\eta)$, we then have

$$(2\beta-2\eta+\gamma-\varepsilon+1)/\theta-[\eta(1-\theta)/\theta^2]\ln(1-\theta+\theta\beta/\eta)-\{[\eta(1-\theta)+2\theta\beta]/\theta^2\}\ln\{[\eta(1-\theta)$$
$$+2\theta\beta+\theta\gamma]/[\eta(1-\theta)+\theta\beta]\}+[(1-\eta)(1-\theta)/\theta^2]\ln[1-\theta+\theta\varepsilon/(1-\eta)]$$
$$+\{(2/3)\gamma^{1/2}/[\eta(1-\theta)+2\theta\beta+\theta\gamma]\}(\varepsilon^{3/2}-\gamma^{3/2}) = G(\beta,\gamma,\varepsilon,\eta) = 0. \tag{30}$$

Associated (Pragerian) field (Fig. 2(e)). By eqn (14) we have

$$\text{I.}\quad \int_{-\eta}^{-\beta}(-u'')\,dt = \int_{-\eta}^{-\beta}(v/S)\,dt = \int_{-\eta}^{-\beta}\{v/[\eta-\theta(\eta+t)]\}\,dt = (-v/\theta)\ln(1-\theta+\theta\beta/\eta). \tag{31}$$

$$\text{II and III.}\quad \int_{-\beta}^{\gamma}(-u'')\,dt = \int_{-\beta}^{\gamma}(v/S)\,dt = \int_{-\beta}^{\gamma}\{v/[\eta(1-\theta)+\theta(2\beta+t)]\}\,dt$$

$$= (v/\theta)\ln\{[\eta+\theta(\gamma+2\beta-\eta)]/[\eta+\theta(\beta-\eta)]\}. \tag{32}$$

$$\text{IV.}\quad \int_\gamma^\varepsilon(-u'')\,dt = \int_\gamma^\varepsilon(v/S)\,dt = v^{1/2}\int_\gamma^\varepsilon t^{-1/2}\,dt = 2v^{1/2}(\varepsilon^{1/2}-\gamma^{1/2}). \tag{33}$$

V. $$\int_{\varepsilon}^{1-\eta}(-u'')\,dt = \int_{\varepsilon}^{1-\eta}(v/S)\,dt = v\int_{\varepsilon}^{1-\eta}[(1-\eta)(1-\theta)+\theta t]^{-1}\,dt$$
$$= (-v/\theta)\ln[1-\theta+\theta\varepsilon/(1-\eta)]. \tag{34}$$

At the ends of the beam (Point E in Fig. 2(e)) concentrated rotations occur in the associated field. Part of this rotation (c) is due to the "concentrated" cost of the clamping moment (p. 141 of Ref. [15] and Ref. [5]) and another part is given by (20b)

$$\Delta = \int_{-\eta}^{-\beta}(-1+vt/S^2)\,dt$$
$$= \int_{-\eta}^{-\beta}\{-1+vt/[\eta-\theta(\eta+t)]^2\}\,dt$$
$$= -\eta+\beta+(v/\theta^2)\{\theta(1-\theta)(\eta-\beta)/[\eta-\theta(\eta-\beta)]+\ln(1-\theta+\theta\beta/\eta)\}. \tag{35}$$

The concentrated rotation at the beam centre (Point F in Fig. 2(e)) is also given by eqn (20a)

$$\delta = \int_{\varepsilon}^{1-\eta}(1-vM/S^2)\,dt = (1-\eta-\varepsilon)-v\int_{\varepsilon}^{1-\eta}\{t/[(1-\eta)(1-\theta)+\theta t]^2\}\,dt$$
$$= 1-\eta-\varepsilon+(v/\theta^2)\{\ln[(1-\theta)+\theta\varepsilon/(1-\eta)]+[\theta(1-\theta)(1-\eta-\varepsilon)]/[(1-\eta)(1-\theta)+\theta\varepsilon]\}. \tag{36}$$

The kinematic admissibility of the associated field implies (by adding the right-hand sides of eqns (31)–(36))

$$-c+(v/\theta)\ln\{[\eta+\theta(\gamma+2\beta-\eta)][\eta(1-\theta+\theta\beta/\eta)^2]^{-1}[1-\theta+\theta\varepsilon/(1-\eta)]^{-1}\}+(1-\theta)(v/\theta)$$
$$\times\{(1-\eta-\varepsilon)/[(1-\eta)(1-\theta)+\theta\varepsilon]+(\eta-\beta)/[\eta-\theta(\eta-\beta)]\}+2\sqrt{v}(\varepsilon^{1/2}-\gamma^{1/2})$$
$$+1-2\alpha+\beta-\varepsilon+(v/\theta^2)\ln\{[1-\theta+\theta\varepsilon/(1-\eta)][1-\theta+\theta\beta/\eta]\}=0. \tag{37}$$

In addition, optimality condition (20c) for the inner R_N^+ region furnishes

$$\int_{-\beta}^{\gamma}\{1-vt/[\eta+\theta(t+2\beta-\eta)]^2\}\,dt = \gamma+\beta-(v/\theta^2)\left\{\ln\left[\frac{\eta+\theta(\gamma+2\beta-\eta)}{\eta+\theta(\beta-\eta)}\right]\right.$$
$$\left.-\frac{\theta[\eta+\theta(2\beta-\eta)](\gamma+\beta)}{[\eta+\theta(\gamma+2\beta-\eta)][\eta+\theta(\beta-\eta)]}\right\}=0. \tag{38}$$

The optimal values of β, γ, ε, η (and v) are given by eqns (27b), (30), (37), (38) (and (26)) and are shown in Fig. 4 for $\theta = 0.8$.

The *total* "*cost*" of the beam is given by

$$\Phi = \int_{-\eta}^{1-\eta}S\,dt+c\eta = (\eta+S_A)(\eta-\beta)/2+(S_A+S_B)(\gamma+\beta)/2$$
$$+[1-\eta+(1-\eta)(1-\theta)+\theta\varepsilon](1-\eta-\varepsilon)/2+\int_{\gamma}^{\varepsilon}\{[(1-\eta)(1-\theta)+\theta\varepsilon]/\sqrt{\varepsilon}\}t^{1/2}\,dt+c\eta$$
$$= -(2/3)[(1-\eta)(1-\theta)+\theta\varepsilon]\sqrt{(\gamma^3/\varepsilon)}+c\eta+2\eta^2+\eta\gamma+1-2\eta-\varepsilon/3+\eta\varepsilon/3$$
$$+\theta[-\eta^2+\beta^2-\eta\gamma+2\beta\gamma+(\gamma^2/2)-(1/2)+\eta+\varepsilon/3-\varepsilon\eta/3+\varepsilon^2/6]. \tag{39}$$

Substituting the optimal values of β, γ, ε, η and v into eqn (39), we obtain the value of Φ_{opt} for Type D solutions (see Fig. 5, $\theta = 0.8$).

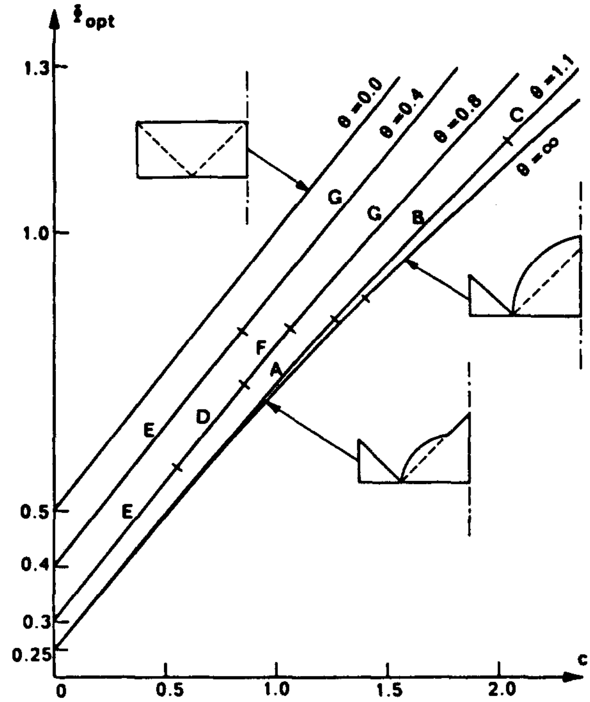

Fig. 4. Optimal values of the parameters α, β, γ, and ε for Types D, E, F and G solutions.

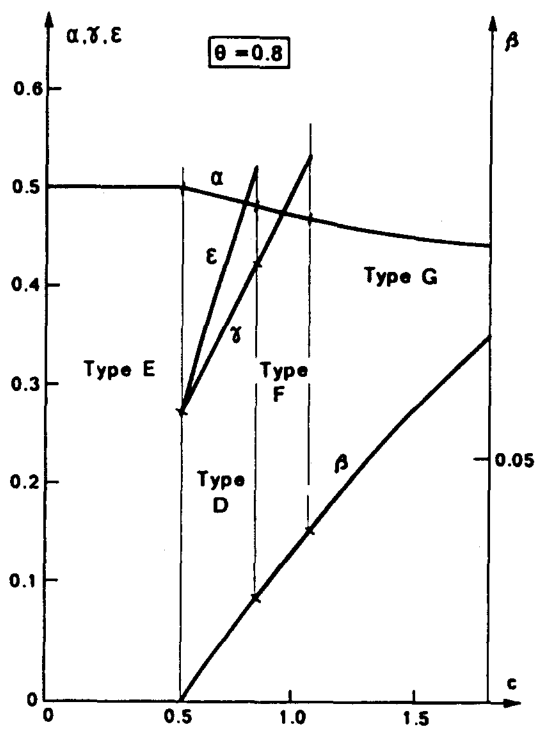

Fig. 5. Variation of the total cost Φ as a function of the support cost factor c and maximum taper θ.

Check by differentiation

Rearranging eqn (27b) in the form $H(\beta, \gamma, \varepsilon, \eta) = 0$ and then incorporating it and eqn (30) into eqn (39) via the Lagrangian multipliers L_1 and L_2, we have

$$\Phi^* = \Phi(\beta, \gamma, \varepsilon, \eta) + L_1 G(\beta, \gamma, \varepsilon, \eta) + L_2 H(\beta, \gamma, \varepsilon, \eta). \tag{40}$$

The usual stationarity conditions $\partial\Phi^*/\partial\beta = 0$, $\partial\Phi^*/\partial\gamma = 0$, $\partial\Phi^*/\partial\varepsilon = 0$, $\partial\Phi^*/\partial\eta = 0$ were determined analytically and, together with eqns (27b) and (30) yielded the optimal values of β, γ, ε, η, L_1 and L_2 which were in complete numerical agreement (ten significant digits) with those furnished by the variational solution. Details of calculations can be found in doctoral theses[25, 26].

Limits of validity. If $\theta \geqslant 1$, then an R_N^- and an R_N^+ region in the Type D solution turn into "fully stressed" R_s regions and hence the solution becomes a Type A one (Fig. 3). One boundary of the set of Type D solutions in Fig. 3 is, therefore, $\theta = 1$.

Other limiting cases are $1-\eta = \varepsilon$ (Fig. 2(d)) which changes the Type D solution into a Type F solution (Fig. 3) and $\varepsilon = \gamma$ which is the limiting case between solutions of Types D and E (Fig. 3). It can be seen from Fig. 4 that at $\varepsilon = \gamma$ we also have $\beta = 0$ and hence the stiffness distribution of the half-beam for Type E solutions is symmetrical.

Type F *solutions* (*Figs* 6(a) *and* (b))

These solutions contain an R_N^-, an R_N^+ and an R_d region and can be derived readily from Type D solutions in which Region V vanishes and hence $\varepsilon = 1-\eta$. For Type F solutions, condition (26) is not valid any more except for the limiting cases between solutions of Types D and F. However, eqns (27a) and (28)–(39) still hold with $\varepsilon = 1-\eta$, although the integral in eqns (29) and (34) and δ in eqn (36) (Fig. 6(b)) take on a zero value. With the above substitution, the optimal values of β, γ, η and v can be readily obtained (Fig. 4).

Check by differentiation. Equation (40) is modified by putting $\varepsilon = 1-\eta$ giving

$$\Phi^* = \Phi(\beta, \gamma, \eta) + L_1 G(\beta, \gamma, \eta). \tag{41}$$

Then the usual stationarity conditions $\partial\Phi/\partial\eta - (\partial\Phi/\partial\beta)(\partial G/\partial\eta)/(\partial G/\partial\beta) = 0$, $\partial\Phi/\partial\gamma - (\partial\Phi/\partial\beta)(\partial G/\partial\gamma)/(\partial G/\partial\beta) = 0$ and eqn (30) with $\varepsilon = 1-\eta$ have been found to numerically confirm the variational solution to ten digit accuracy.

Limiting cases of Type F *solution*. Along the following limiting cases Type F solutions change to another type

Type F/G	$\gamma = 1-\eta$
Type F/D	$S(t)\|_{t=1-\eta} = 1-\eta$
Type F/B	$\theta = 1$.

It has been checked that along these limiting cases the equations for both types of adjacent solutions reduce to the same set of expressions.

Type G solutions

This type of solution is characterized by a single R_N^- and a single R_N^+ region (Figs 6(c) and (d)). The necessary equations are modified from Type D solutions by substituting $\gamma = \varepsilon = 1-\eta$ into eqns (30), (37) and (38).

Check by differentiation. Equation (41) is further modified by putting $\gamma = \varepsilon = 1-\eta$, which furnishes

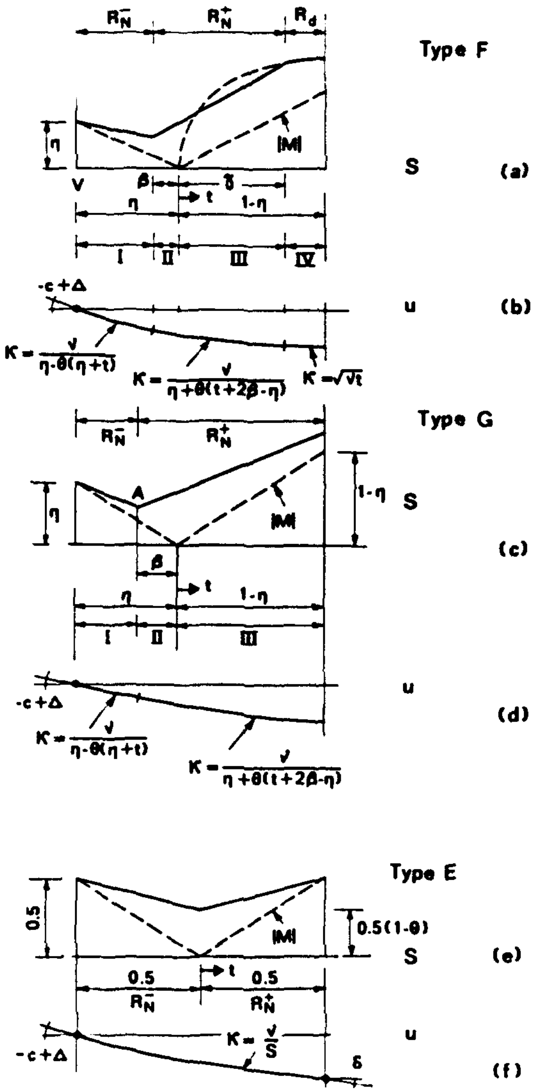

Fig. 6. Types F, G and E solutions.

$$\Phi^* = \Phi(\beta, \eta) + L_1 G(\beta, \eta). \tag{42}$$

The stationarity condition $(\partial\Phi/\partial\beta)(\partial G/\partial\eta) - (\partial\Phi/\partial\eta)(\partial G/\partial\beta) = 0$ and eqn (30) with $\gamma = \varepsilon = 1-\eta$ confirmed the optimal values η, β obtained from the variational solution.

Limiting case between Types G *and* E *solutions*. Equation (30) with $\beta = 0$ furnishes

$$\theta(1-2\eta) - \eta(1-\theta) \ln [(\eta - 2\theta\eta + \theta)/\eta] = 0 \tag{43}$$

which is clearly satisfied by $\eta = 0.5$ for *any* θ-value. Moreover, eqns (37) and (38) with $\beta = 0$, $\eta = 0.5$ and $\gamma = \varepsilon = 1-\eta$ reduce to

$$\ln [1/(1-\theta)] - \theta - \theta^2/2v = 0, \quad v/\theta^2 = -\{2[\theta + \ln (1-\theta)]\}^{-1} \tag{44}$$

$$v/\theta^2 = (1/2 + c)/[\theta + (1-2\theta) \ln (1-\theta)]. \tag{45}$$

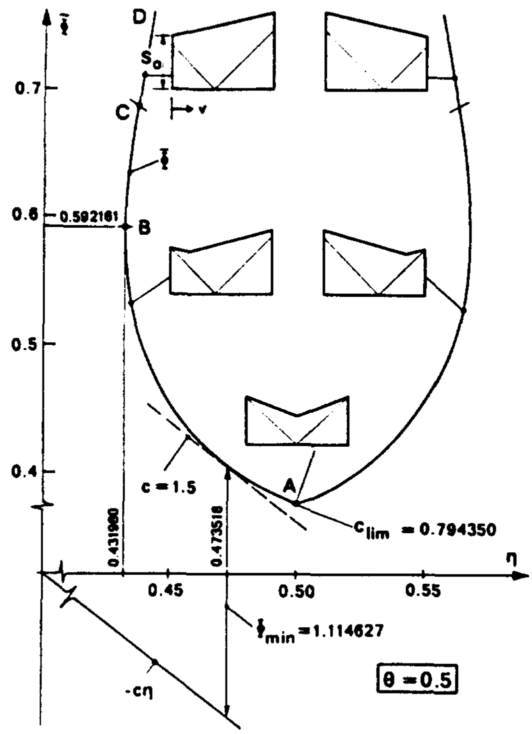

Fig. 7. Independent derivation of optimal solutions.

Equating the right-hand sides of eqns (44) and (45) we have

$$c = [(\theta-1)\ \ln\ (1-\theta)-\theta]/[\theta+\ln\ (1-\theta)] \tag{46}$$

which is the equation for the boundary of Types E and G solutions in Fig. 3. Along the same boundary (35) with $\eta = 0.5$, $\beta = 0$, $\gamma = \varepsilon = 1-\eta$ and eqns (44) give $\Delta = -1$ for *any* θ-value.

Check on Types G *and* E *solutions by direct minimization.* The total cost $\Phi(\beta, \eta)$ in eqn (42) can be split into two components: beam cost $\bar{\Phi}$ and cost of clamping moment ηc. The former ($\bar{\Phi}$) is independent of c and is shown in Fig. 7 for $\theta = 0.5$ and various η values, taking the kinematic admissibility requirement $G(\eta, \beta) = 0$ in eqn (42) into consideration. It will be seen that (i) the total cost Φ is given in Fig. 7 by the vertical distance between the graphs of $-c\eta$ and $\bar{\Phi}$ and (ii) the optimal value of η can be obtained by drawing a tangent to $\bar{\Phi}$ which is parallel to the graph of $-c\eta$. This procedure is illustrated in Fig. 7 for $c = 1.5$. At Point A the slope of $\bar{\Phi}$ is $d\bar{\Phi}/d\eta = 0.79434972$. This means that for any c-value which is smaller than the above slope value Point A represents the optimal solution which is of Type E with $\eta = 0.5$ and by eqn (43) $\beta = 0$. The validity of this unique value of η for a range of c-values is due to the nonuniqueness of the slope of $\bar{\Phi}$ at Point A. It can also be seen from Fig. 7 that a Type G solution is valid for an indefinitely large value of c because the slope of $\bar{\Phi}(\eta)$ reaches infinity at Point B. By raising the stiffness over the R_N^+ region further, we can extend the $\bar{\Phi}$-curve to Points C and D. Over the segment CD the compatibility conditions become

$$\int_0^1 [(\eta-v)/(S_0+\theta v)]\ dv = 0, \quad \eta = [1/\ln\ (1+\theta/S_0)]-S_0/\theta. \tag{47}$$

Figure 7 therefore indicates that there exists only one local minimum in this problem which is also the global minimum of Φ.

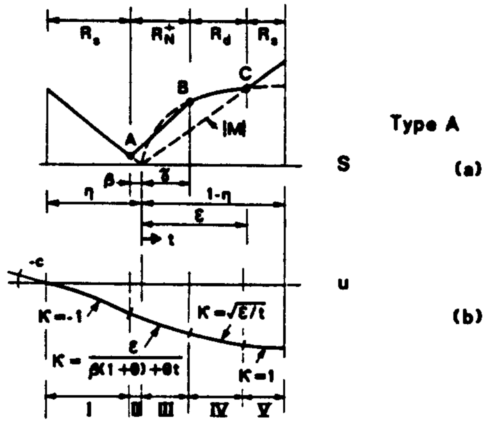

Fig. 8. Type A solutions.

Type E *solutions* (*Fig.* 6(e))

Reasons for the fact that the stiffness distribution for this type is independent of c and θ were explained in the last section (Fig. 7) which was based on direct minimization. The variational formulation for the same type of solution is discussed herein.

Associated field (*Fig.* 6(f)). From eqns (20a), (20b) and (14) we have

$$\delta = \int_0^{0.5} (1 - vt/S^2)\, dt = (1/2) + (v/\theta^2)\,[\theta + \ln\,(1-\theta)]$$

$$\Delta = \int_{-0.5}^{0} (1 - vt/S^2)\, dt = (-1/2) + (v/\theta^2)\,[\theta + \ln\,(1-\theta)]$$

$$\int_{-0.5}^{0.5} (-u'')\, dt = \int_{-0.5}^{0.5} (v/S)\, dt = (-2v/\theta)\,\ln\,(1-\theta). \tag{48}$$

Then kinematic admissibility requires (Fig. 6(f))

$$-c + \frac{2v}{\theta^2}[\theta + (1-\theta)\,\ln\,(1-\theta)] = 0$$

or

$$v = \frac{c\theta^2}{2}\Big/[\theta + (1-\theta)\,\ln\,(1-\theta)]. \tag{49}$$

Check on Type E/G *limiting case*. Since for Type G solutions the concentrated rotation at the beam centre must be zero, along the Type E/G limiting case we have in eqns (48) $\delta = 0$, $\Delta = -1$. Substituting the v-value from eqns (49) into the equation for δ in eqn (48) we obtain eqn (46) which constitutes an independent confirmation of the same result from two different types of solutions.

Type A *solutions* (*Figs* 8(a) *and* (b))

If in the Niordson constraint $\theta \geqslant 1.0$ then the moment diagram (stress constraint) governs the solution over finite beam lengths. Type A solutions consist of two R_s, one R_d and one R_N^+ region. The stiffnesses at points A, B and C are obtained from purely geometrical considerations (Fig. 8(b))

$$S_A = \beta, \quad S_B = \beta(1+\theta)+\gamma\theta, \quad S_C = \varepsilon. \tag{50}$$

As over the R_d region (IV in Fig. 8(c)) eqns (16) give $S(t) = \sqrt{(vt)}$, eqns (50) imply

$$\sqrt{(v\varepsilon)} = \varepsilon, \quad v = \varepsilon \tag{51}$$

$$\sqrt{(v\gamma)} = \beta(1+\theta)+\gamma\theta, \quad v = [\beta(1+\theta)+\gamma\theta]^2/\gamma. \tag{52}$$

Then from eqns (51) and (52) we have

$$\varepsilon = [\beta(1+\theta)+\gamma\theta]^2/\gamma. \tag{53}$$

Elastic kinematic admissibility

I. $$(R_s) \quad \int_{-\eta}^{-\beta} (M/S)\, dt = \int_{-\eta}^{-\beta} dt = \beta-\eta \tag{54}$$

II, III. $$(R_N^+) \quad \int_{-\beta}^{\gamma} (M/S)\, dt = \int_{-\beta}^{\gamma} \{t/[\beta(1+\theta)+\theta t]\}\, dt$$
$$= (\gamma+\beta)/\theta-[\beta(1+\theta)/\theta^2] \ln\, [1+\theta+\theta\gamma/\beta] \tag{55}$$

IV. $$\int_{\gamma}^{\varepsilon} (M/S)\, dt = (1/\sqrt{v}) \int_{\gamma}^{\varepsilon} \sqrt{t}\, dt = (2/3)\,(\sqrt{(\gamma\varepsilon^3)}-\gamma^2)/[\beta(1+\theta)+\gamma\theta] \tag{56}$$

V. $$\int_{\varepsilon}^{1-\eta} (M/S)\, dt = \int_{\varepsilon}^{1-\eta} dt = 1-\eta-\varepsilon. \tag{57}$$

Then the left-hand side of eqn (21a) is given by the sum of the right-hand sides of eqns (54)–(57) which is denoted by $G(\beta, \gamma, \varepsilon, \eta)$

$$G(\beta, \gamma, \varepsilon, \eta) = 1-2\eta+\beta-\varepsilon+(\gamma+\beta)/\theta-[(1+\theta)\beta/\theta^2] \ln\, (1+\theta+\theta\gamma/\beta)$$
$$+(2/3)\,(\sqrt{(\gamma\varepsilon^3)}-\gamma^2)/[\beta(1+\theta)+\gamma\theta] = 0. \tag{58}$$

Associated kinematic admissibility. Using eqns (13) and (14), we have

I. $$\int_{-\eta}^{-\beta} (-u'')\, dt = \int_{-\eta}^{-\beta} dt = \beta-\eta \tag{59}$$

II, III. $$\int_{-\beta}^{\gamma} (v/S)\, dt = v\int_{-\beta}^{\gamma} \{1/[\beta(1+\theta)+\theta t]\}\, dt$$
$$= (v/\theta)\, \ln\, (1+\theta+\theta\gamma/\beta) \tag{60}$$

IV. $$\int_{\gamma}^{\varepsilon} (v/S)\, dt = \sqrt{v}\int_{\gamma}^{\varepsilon} t^{-1/2}\, dt = 2\sqrt{v}(\sqrt{\varepsilon}-\sqrt{\gamma}) \tag{61}$$

V. $$\int_{\varepsilon}^{1-\eta} (-u'')\, dt = 1-\eta-\varepsilon. \tag{62}$$

Then from the condition

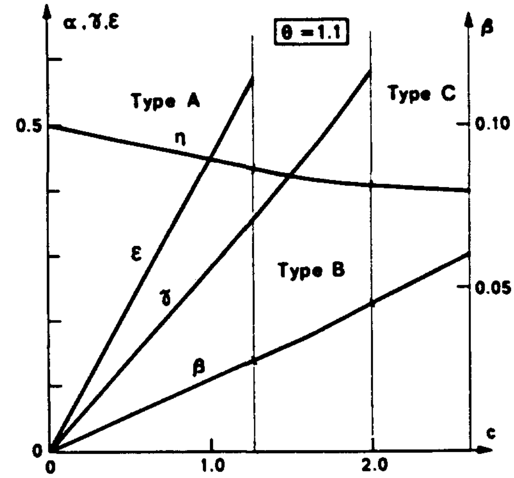

Fig. 9. Optimal values of the parameters α, β, γ, and ε for Types A, B and C solutions.

$$\int_{-\eta}^{1-\eta} (-u'')\,\mathrm{d}t = 0$$

and eqns (59)–(62) we have

$$-c+1-2\eta+\beta-\varepsilon+(v/\theta)\ \ln\,(1+\theta+\theta\gamma/\beta)+2\sqrt{v}(\sqrt{\varepsilon}-\sqrt{\gamma}) = 0. \tag{63}$$

Moreover, from eqns (20c) we have

$$\int_{\beta}^{\gamma} (1-vM/S^2)\,\mathrm{d}t = \int_{-\beta}^{\gamma} \{1-vt/[\beta(1+\theta)+\theta t]^2\}\,\mathrm{d}t$$

$$= \gamma+\beta-(v/\theta^2)\,\{1+\theta-(1+\theta)\beta/[(1+\theta)\beta+\gamma\theta]-\ln\,(1+\theta+\theta\gamma/\beta)\} = 0. \tag{64}$$

The optimal values of $\eta, \beta, \gamma, \varepsilon$ and v were obtained from eqns (51), (52), (58), (63) and (64) (Fig. 9).

The *total cost* of the beam is given by (Fig. 8(b))

$$\Phi = \int_{-\eta}^{1-\eta} S\,\mathrm{d}t+\eta c = (1/2)\,[\eta^2+(1-\eta)^2-\varepsilon^2+\beta^2+2\gamma\beta$$

$$+\theta(\beta+\gamma)^2]+(2/3)\,(\varepsilon^2-\sqrt{(\gamma^3\varepsilon)})+\eta c. \tag{65}$$

Check by differentiation. Substituting eqn (53) into eqns (58) and (65), the modified cost can be expressed as

$$\Phi^* = \Phi(\eta,\beta,\gamma)+L_1G(\eta,\beta,\gamma). \tag{66}$$

The usual stationarity conditions then confirmed the results obtained by the variational approach.

Types B *and* C *solutions*

These can be obtained by modifying Type A solutions through the substitutions $\varepsilon = 1-\eta$ and $\gamma = \varepsilon = 1-\eta$, respectively. For these two cases, respectively, eqns (52), (54) and eqns (52)–(54) are not valid.

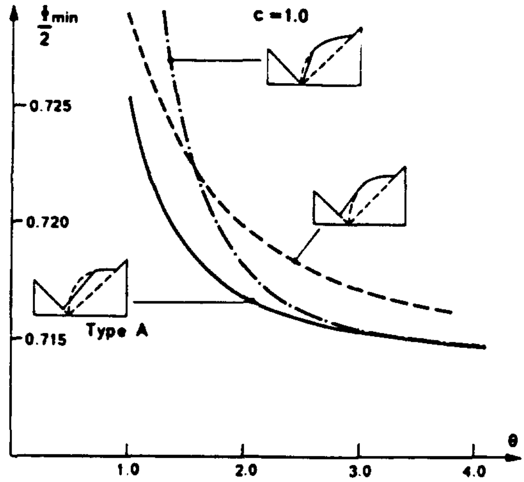

Fig. 10. Confirmation of non-optimality of certain types of solutions.

Finally Fig. 10 shows a cost comparison of optimal Type A solutions (continuous line) and optimal solutions within the constraint that (i) the stiffness function in the R_N^+ region must be tangential to that in the R_d region (broken line) and (ii) $S(t)$ at $t = 0$ is zero (dash-dot line). It can be seen that as predicted by the variational formulation, the former is always more economical.

CONCLUSIONS

(a) It has been demonstrated that static-kinematic optimality criteria, originally proposed for plastic design, can also handle elastic design with deflection, stress and Niordson constraints as well as limits on minimum and maximum cross-sectional area.

(b) Considering a built-in beam with stress and Niordson constraints, it was found that the solutions often contain understressed (R_d) regions and the types of solutions can take on a number of different forms (Fig. 3).

(c) As in plastic design with Niordson constraints[5], the associated curvature field often contains curvature impulses (concentrated rotations).

(d) Whereas in optimal plastic design only the associated field had to be considered, optimal elastic design involves kinematic admissibility of both elastic and associated curvatures. In the case of Niordson constraints, additional optimality conditions must be fulfilled.

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